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WKB (Liouville–Green) Analysis of Second Order Difference Equations and Applications

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A discrete analog of the WKB or Liouville–Green approximation for the solution of second order differential equations is discussed. The method is then applied to obtain asymptotic solutions to second order difference equations whose coefficients are regularly varying.

1. INTRODUCTION

The WKB (Wentzell, Kramers, and Brillouin) or Liouville–Green method for second order differential equations is a powerful method for obtaining asymptotic approximations to solutions of these types of equations (Olver [16]). Here motivated by the work of Braun [1] we present a discrete analog of this method (Smith [18], Wilmott [24]). Consider the second order difference equation

$$d(n+1)\, y(n+1) - q(n)\, y(n) + y(n-1) = 0, \tag{1.1}$$

where $d(n)$ and $q(n)$ are sequences of complex numbers with $d(n) \neq 0$, $n = 1, 2, \ldots$. If we look for a solution of (1.1) of the form

$$y(n) = \prod_{k=n_0}^{n} u(k), \qquad n > n_0, \tag{1.2}$$

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we arrive at the discrete Riccati equation

$$d(n+1)\,u(n+1)\,u(n)-q(n)\,u(n)+1=0. \tag{1.3}$$

If we rewrite Eq. (1.3) as

$$d(n+1)\,\frac{u(n+1)}{u(n)}\,u(n)^2-q(n)\,u(n)+1=0, \tag{1.4}$$

and suppose that $\lim_{n\to\infty} d(n+1)=1$ and $\lim_{n\to\infty} u(n)/u(n+1)=1$, then the solutions of the equation

$$u_0(n)^2-q(n)\,u_0(n)+1=0, \tag{1.5}$$

i.e.,

$$u_0(n)=\frac{q(n)+\sqrt{q(n)^2-4}}{2}, \tag{1.6}$$

and

$$v_0(n)=\frac{q(n)-\sqrt{q(n)^2-4}}{2}, \tag{1.7}$$

where as a complex function we take the branch of the square root so that

$$|z+\sqrt{z^2-4}|>1 \tag{1.8}$$

for $z\in C\backslash[-2,2]$, might be expected to yield adequate approximations to (1.4). This is indeed true if the coefficients in (1.4) approach sufficiently rapidly their asymptotic values (Máté and Nevai [11], Van Assche and Geronimo [23], Máté, Nevai, and Totik [12]). If we suppose u_n/u_{n+1} to be known and solve (1.4) for u_n then

$$\begin{aligned} u(n)=&\frac{q(n)}{2}\frac{u(n)}{d(n+1)\,u(n+1)}\\ &+\sqrt{\left(\frac{q(n)}{2}\frac{1}{d(n+1)}\frac{u(n)}{u(n+1)}\right)^2-\frac{u(n)}{d(n+1)\,u(n+1)}}. \end{aligned} \tag{1.9}$$

We now follow Braun and develop an iteration procedure for (1.9) by replacing $u(n)$ in the right hand side of that equation with $u_0(n)$ which yields

$$u_1(n)=\frac{q(n)}{2}\frac{1}{d(n+1)}\frac{u_0(n)}{u_0(n+1)}$$
$$+\sqrt{\left\{\frac{q(n)}{2}\frac{u_0(n)}{d(n+1)\,u_0(n+1)}\right\}^2-\frac{u_0(n)}{d(n+1)\,u_0(n+1)}}.$$

Since $u_0(n)/d(n+1)\,u_0(n+1)\approx 1$ we expand the left hand side in a Taylor series in

$$t=\frac{u_0(n)}{d(n+1)\,u_0(n+1)} \tag{1.10}$$

about 1. This gives

$$u_1(n)=u_0(n)+\left\{\frac{q(n)}{2}+\frac{2(q(n)/2)^2-1}{2\sqrt{(q(n)/2)^2-1}}\right\}(t-1)+O((t-1)^2). \tag{1.11}$$

This equation can be written using the definition of $u_0(n)$ as

$$u_1(n)=u_0(n)\left\{1+\frac{u_0(n)}{d(n+1)\,u_0(n+1)}\frac{[u_0(n)-d(n+1)\,u_0(n+1)]}{u_0(n)-1/u_0(n)}\right\}$$
$$+O((t-1)^2). \tag{1.12}$$

Since $u_0(n)/d(n+1)\,u_0(n+1)-1=t-1$ a useful solution would be to replace $u_0(n)/d(n+1)\,u_0(n+1)$ by 1 in the above equation. This we will do, however, for the applications below it will be more convenient to consider

$$u_2(n)=u_0(n)\left\{1-\frac{[u_0(n)-d(n+1)\,u_0(n+1)]}{u_0(n)-1/u_0(n)}\right\}^{-1}. \tag{1.13}$$

This is one of the approximate solutions of the Riccati equation that we wish to use. If we repeat the same procedure with $v_0(n)$ in place of $u_0(n)$ we find that the second approximate solution is

$$v_2(n)=v_0(n)\left\{1-\frac{[v_0(n)-d(n+1)\,v_0(n+1)]}{v_0(n)-1/v_0(n)}\right\}^{-1}. \tag{1.14}$$

We note that the procedure given above, although purely formal, is useful in obtaining approximate solutions. One might imagine cases that would arise where higher order terms would be useful, but in the next section we give conditions on the coefficients in (1.1) which imply that we need not take more terms into consideration. Finally in Section 3 we use this technique to obtain asymptotics for polynomials orthogonal on an infinite interval whose recurrence coefficients are regularly or slowly varying. The idea of explicitly using the theory of regular variation to compute the asymptotics of orthogonal polynomials was introduced by Van Assche [19, 20] although precursors can be found in Nevai and Dehesa [13]. There has been much interest in polynomials whose recurrence coefficients are regularly varying because of their relation to polynomials orthogonal with respect to Freud type weights (Lubinsky [7], Lubinsky and Saff [10], Lubinsky, Maskar, and Saff [9], Nevai [14, 15], Rakhmanov [17], Van Assche [22]). In Van Assche and Geronimo [21] and Geronimo and Van Assche [4] strong asymptotics outside the oscillatory region were given for polynomials whose recurrence coefficients are regularly varying. Here and in future paper (Geronimo, Smith, and Van Assche [5]) we extend the results in [21] to a wider class of polynomials, in particular to those which have recurrence coefficients that are slowly varying. Polynomials with slowly varying recurrence coefficients are closely related to those orthogonal with respect to Erdös-type weights which have been extensively investigated by Lubinsky [8].

2. Justification of the Approximate Solutions

In order to get conditions on the coefficients in (1.1) so that $u_2(n)$ and $v_2(n)$ can be used to approximate the solutions of (1.1) we begin by considering the equation

$$s(n)\, y(n+1) + r(n)\, y(n) + y(n-1) = 0, \tag{2.1}$$

where $r(n)$ and $s(n)$ are sequences of complex numbers and $s(n) \neq 0$ for n sufficiently large.

Theorem 2.1. *Suppose for $N_1 > n \geqslant N$ (N_1 may be infinite) there are solutions $f(n)$ and $g(n)$, respectively, of the discrete Riccati equation*

$$s(n)\, f(n+1)\, f(n) + r(n)\, f(n) + 1 = \xi(n), \tag{2.2}$$

$$s(n)\, g(n+1)\, g(n) + r(n)\, g(n) + 1 = \zeta(n), \tag{2.3}$$

where $s(n)\neq 0$, $g(n)\neq 0$, *and* $f(n)\neq 0$, $N_1>n\geqslant N$,

$$\sum_{i=N}^{N_1-1} |\xi(i)|<\infty, \tag{2.4a}$$

$$\sum_{i=N}^{N_1-1} |\zeta(i)|<\infty, \tag{2.4b}$$

and there exists a constant c such that

$$\left|\sum_{k=i}^{n-1} \sum_{j=i}^{k} (s(j)\,f(j+1)\,f(j))^{-1}\right|<c, \qquad N_1>n\geqslant i\geqslant N, \tag{2.5a}$$

and

$$\left|\sum_{k=n+1}^{i} \sum_{j=k}^{i} s(j)\,g(j+1)\,g(j)\right|<c, \qquad N_1>i>n\geqslant N. \tag{2.5b}$$

Then there exist solutions $y_\pm$ *of* (2.1) *such that*

$$\left|y_+(n)\prod_{i=N}^{n} f(i)^{-1}-1\right|\leqslant \exp\left\{c\sum_{j=N}^{n}|\xi(j)|\right\}-1, \qquad N_1>n\geqslant N, \tag{2.6a}$$

and

$$\left|y_-(n)\prod_{i=N}^{n} g(i)^{-1}-1\right|\leqslant \exp\left\{c\sum_{j=n+1}^{N_1-1}|\zeta(j)|\right\}-1, \qquad N_1>n\geqslant N. \tag{2.6b}$$

Proof. Substitute

$$y_+(n)=\left(\prod_{i=N}^{n} f(i)\right)(1+\phi(n)) \tag{2.7}$$

in Eq. (2.1) then use (2.2) to find

$$s(n)\,f(n+1)\,f(n)\,\phi(n+1)+f(n)\,r(n)\,\phi(n)+\phi(n-1)=-\xi(n). \tag{2.8}$$

Now use (2.2) once again to eliminate $f(n)\,r(n)$ in the above equation, which gives

$$s(n)\,f(n+1)\,f(n)(\phi(n+1)-\phi(n))-(\phi(n)-\phi(n-1))=-\xi(n)(1+\phi(n)). \tag{2.9}$$

Two linearly independent solutions of the homogeneous equation

$$s(n)\,f(n+1)\,f(n)(\phi^0(n+1)-\phi^0(n))-(\phi^0(n)-\phi^0(n-1))=0 \tag{2.10}$$

are

$$\phi_1^0(n) = 1, \tag{2.11}$$

and

$$\phi_2^0(n) = \sum_{i=N-1}^{n-1} \prod_{j=N}^{i} (s(j)\, f(j+1)\, f(j))^{-1}, \qquad n \geqslant N. \tag{2.12}$$

If we use the method of variation of constants we find that

$$\phi(n) = -\sum_{i=N}^{n-1} \xi(i)(1+\phi(i)) \times \frac{(\phi_1^0(i)\,\phi_2^0(n) - \phi_2^0(i)\,\phi_1^0(n))}{s(i)\, f(i+1)\, f(i)(\phi_2^0(i+1)\,\phi_1^0(i) - \phi_1^0(i+1)\,\phi_2^0(i))}, \qquad n \geqslant N \tag{2.13}$$

is a solution of (2.9). Equations (2.11) and (2.12) allow (2.13) to be rewritten as

$$\phi(n) = -\sum_{i=N}^{n-1} G(n, i)\, \xi(i)(1+\phi(i)), \tag{2.14}$$

where

$$G(n, i) = \sum_{k=i}^{n-1} \prod_{j=i}^{k} (s(j)\, f(j+1)\, f(j))^{-1}. \tag{2.15}$$

Set

$$\phi^l(n) = -\sum_{i=N}^{n-1} G(n, i)\, \xi(i)(1+\phi^{l-1}(i)) \tag{2.16}$$

with $\phi^0(n) = 0$. Then

$$|\phi^1(n)| = |\phi^1(n) - \phi^0(n)| \leqslant c \sum_{i=N}^{n-1} |\xi(i)|, \tag{2.17}$$

by (2.5a). Since

$$\phi^l(n) - \phi^{l-1}(n) = -\sum_{i=N}^{n-1} G(n, i)\, \xi(i)(\phi^{l-1}(i) - \phi^{l-2}(i)), \tag{2.18}$$

we find by counting that

$$|\phi^l(n) - \phi^{l-1}(n)| \leqslant \frac{(c \sum_{i=N}^{n-1} |\xi(i)|)^l}{l!}, \tag{2.19}$$

which in turn implies (2.6a) because $\phi(n)=\sum_{l=1}^{\infty}(\phi^l(n)-\phi^{l-1}(n))$. To obtain (2.6b) make the substitution

$$y_-(n)=\prod_{i=N}^{n} g(i)(1+\hat{\phi}(n)) \tag{2.20}$$

in (2.1), then follow procedures similar to those above, with the exception that instead of (2.12) use

$$\hat{\phi}_2^0(n)=\sum_{k=n+1}^{N_1-1}\prod_{j=k}^{N_1-1} s(j)\,g(j+1)\,g(j), \tag{2.21}$$

which yields

$$\hat{\phi}(n)=\sum_{i=n+1}^{N_1-1}\zeta(i)\,G^1(n,i)(1+\hat{\phi}(i)), \tag{2.22}$$

where

$$G^1(n,i)=\sum_{k=n+1}^{i}\prod_{j=k}^{i-1} s(j)\,g(j+1)\,g(j). \tag{2.23}$$

Now using successive approximations, (2.4b) and (2.5b) yield (2.6b).

LEMMA 2.2. *Suppose in* (2.2) *that* $f(n)\neq 0$ *and* $s(n)\neq 0$ *for* $n\geqslant N$ *and let* $y(n)$ *be a solution of* (2.1) *with initial conditions*

$$y(N-1)=1, \tag{2.24}$$

$$y(N)=h. \tag{2.25}$$

Then

$$y(n)=\prod_{i=N}^{n} f(i)(1+\phi(n)), \tag{2.26}$$

with

$$\phi(n)=\left(\frac{h}{f(N)}-1\right)\phi_2^0(n)-\sum_{i=N}^{n-1}\xi(i)(1+\phi(i))\,G(n,i),\qquad n\geqslant N-1, \tag{2.27}$$

where it is always assumed that the empty product is equal to one.

Proof. From (2.12) we find that $\phi_2^0(N-1)=0$ and $\phi_2^0(N)=1$. Set

$$y(n)=\prod_{i=N}^{n} f(i)(1+\varphi(i)); \tag{2.28}$$

then $\phi(n)$ is a solution of (2.29) with initial conditions $\phi(N-1)=0$ and $\phi(N)=(h/f(N))-1$. Using (2.11) and (2.12) we find that

$$\phi(n)=c_1\phi_1^0(n)+c_2\phi_2^0(n)-\sum_{i=1}^{n-1}\xi(i)(1+\phi(i))G(n,i), \qquad n\geqslant N-1, \tag{2.29}$$

where $G(n, i)$ is given by (2.15). The boundary conditions on $\phi(n)$ imply that $c_1=0$ and $c_2=\phi(N)$, which yield the result.

The rest of this section is devoted to finding conditions on $d(n)$ and $q(n)$ so that the hypotheses of Theorem 2.1 are fulfilled.

THEOREM 2.3. *In* (1.1) *let* $q(n)=q(x,n)$, $N\leqslant n<N_1$, x *complex, be a complex function of* x *which is finite for* x *finite. Suppose* $d(n)$ *is a sequence of complex numbers such that* $d(n)\neq 0$, $N\leqslant n<N_1$. *Suppose there exists a compact set* C *of the complex plane containing an open set* U *such that* $[-2,2]\subset U$ *and* $q(x,n)\notin C$ *for all* $N\leqslant n<N_1<\infty$. *Finally suppose*

$$\frac{u_0(x,n)-d(n+1)\,u_0(x,n+1)}{u_0(x,n)-1/u_0(x,n)}\neq 1, \qquad N\leqslant n<N_1, \tag{2.30}$$

and

$$\frac{v_0(x,n)-d(n+1)\,v_0(x,n+1)}{v_0(x,n)-1/v_0(x,n)}\neq 1, \qquad N\leqslant n<N_1. \tag{2.31}$$

Then there exist two solutions $y_\pm$ *of* (2.1) *such that*

$$\left|y_+(n)\prod_{i=N}^{n}u_2(i)^{-1}-1\right|\leqslant\exp\left\{c\sum_{j=N}^{n}|\xi(j)|\right\}-1, \qquad N_1>n\geqslant N, \tag{2.32a}$$

and

$$\left|y_-(n)\prod_{i=N}^{n}v_2(i)^{-1}-1\right|\leqslant\exp\left\{c\sum_{j=n+1}^{N_1-1}|\zeta(j)|\right\}-1, \qquad N_1>n\geqslant N, \tag{2.32b}$$

where

$$\xi(n)=d(n+1)\,u_2(n+1)\,u_2(n)-q(x,n)\,u_2(n)+1 \tag{2.33a}$$

and

$$\zeta(n)=d(n+1)\,v_2(n+1)\,v_2(n)-q(x,n)\,v_2(n)+1. \tag{2.33b}$$

For N_1 infinite assume that (2.30) *and* (2.31) *hold for all $n \geqslant N$, that*

$$q(x, n) \notin C \qquad \forall n \geqslant N, \tag{2.34}$$

and that

$$\begin{aligned} \sum_N^\infty (|\Delta(n)|^2 + |r(n)|^2) < \infty, \\ \sum_N^\infty (|\Delta(n+1) - \Delta(n)| + |r(x, n+1) - r(x, n)|) < \infty, \end{aligned} \tag{2.35}$$

where

$$\Delta(n) = 1 - d(n) \tag{2.36}$$

and

$$r(x, n) = q(x, n+1) - q(x, n). \tag{2.37}$$

Then

$$\sum_N^\infty |\xi(n)| < \infty \qquad \text{and} \qquad \sum_N^\infty |\zeta(n)| < \infty \tag{2.38}$$

and (2.32) *holds for all n, $N \leqslant n < \infty$.*

Before proving the theorem we prove a technical lemma.

LEMMA 2.4. *Let $u_0(x, n)$ and $v_0(x, n)$ be given by* (1.6) *and* (1.7), *respectively, and set*

$$s(x, n) = \sqrt{q(x, n+1)^2 - 4} + \sqrt{q(x, n)^2 - 4}. \tag{2.39}$$

Then

$$u_0(x, n+1) - u_0(x, n) = \frac{r(x, n)}{2}\left[1 + \frac{q(x, n+1) + q(x, n)}{s(x, n)}\right], \tag{2.40}$$

$$v_0(x, n+1) - v_0(x, n) = \frac{r(x, n)}{2}\left[1 - \frac{q(x, n+1) + q(x, n)}{s(x, n)}\right], \tag{2.41}$$

$$u_0(x, n+1) - 2u_0(x, n) + u_0(x, n-1)$$
$$= \frac{[r(x, n) - r(x, n-1)]}{2}\left[1 + \frac{q(x, n+1) + q(x, n)}{s(x, n)}\right]$$
$$+ \frac{r(x, n-1)}{2} \frac{[r(x, n) + r(x, n-1)]}{s(x, n)}$$
$$- \frac{r(x, n-1)(q(x, n) + q(x, n-1))}{2s(x, n)\, s(x, n-1)}$$
$$\times \left[\frac{q(x, n+1)^2 - q(x, n-1)^2}{\sqrt{q(x, n+1)^2 - 4} + \sqrt{q(x, n-1)^2 - 4}}\right], \tag{2.42}$$

and

$$[v_0(x, n+1) - 2v_0(x, n) + v_0(x, n-1)]$$
$$= \frac{r(x, n) - r(x, n-1)}{2}\left[1 - \frac{q(x, n+1) + q(x, n)}{s(x, n)}\right]$$
$$- \frac{r(x, n-1)}{2}\left[\frac{r(x, n) + r(x, n-1)}{s(x, n)}\right]$$
$$+ \frac{r(x, n-1)(q(x, n) + q(x, n-1))}{2s(x, n)\, s(x, n-1)}$$
$$\times \left[\frac{q(x, n+1)^2 - q(x, n-1)^2}{\sqrt{q(x, n+1)^2 - 4} + \sqrt{q(x, n-1)^2 - 4}}\right]. \tag{2.43}$$

Proof. Equations (2.40) and (2.41) follow immediately from the definitions of u_0, v_0, r, and s. To show (2.42) subtract (2.40) from itself with n replaced by $n-1$ to find

$$u_0(x, n+1) - 2u_0(x, n) + u_0(x, n-1)$$
$$= \frac{r(x, n) - r(x, n-1)}{2}\left[1 + \frac{q(x, n+1) + q(x, n)}{s(x, n)}\right]$$
$$+ \frac{r(x, n-1)}{2}\left[\frac{r(x, n) + r(x, n-1)}{s(x, n)}\right]$$
$$+ \frac{r(x, n-1)}{2}(q(x, n) + q(x, n-1))\left[\frac{1}{s(x, n)} - \frac{1}{s(x, n-1)}\right]. \tag{2.44}$$

If we put the last term on the right hand side of the above equation over a common denominator, then clear the radicals in the numerator, we arrive at (2.42). Equation (2.43) follows in an analogous fashion.

Proof of Theorem 2.3. We begin by noting that the hypothesis on $q(x, n)$ imply that there exists a $d>1$ such that $|u_0(x, n)|>d$ and $|v_0(x, n)|<d^{-1}$ for $N \leqslant n < N_1$. This coupled with the fact that $d(n)$ is nonzero for $N \leqslant n < N_1$ implies that $u_2(x, n)$ and $v_2(x, n)$ are bounded away from zero. Equations (2.30) and (2.31) imply that they are bounded away from infinity. If N_1 is finite this is sufficient for the conclusions of the theorem to hold since all the sums and products in (2.4a), (2.4b), (2.5a), and (2.5b) are finite. Suppose N_1 is infinite; then (2.35) implies that $\lim_{n \to \infty} d(n)=1$. If we write (1.13) and (1.14) as

$$u_2(x, n)=u_0(x, n)\{1-\gamma(x, n)\}^{-1} \tag{2.45}$$

and

$$v_2(x, n)=v_0(x, n)\{1-\omega(x, n)\}^{-1}, \tag{2.46}$$

where

$$\gamma(x, n)=\frac{[u_0(x, n)-d(n+1)\, u_0(x, n+1)]}{u_0(x, n)-1/u_0(x, n)} \tag{2.47}$$

and

$$\omega(x, n)=\frac{[v_0(x, n)-d(n+1)\, v_0(x, n+1)]}{v_0(x, n)-1/v_0(x, n)}, \tag{2.48}$$

we see that $\gamma(x, n)$ and $\omega(x, n)$ can be recast as

$$\gamma(x, n)=\frac{(u_0(x, n+1)\, \Delta(n+1)+d(n+1)(u_0(x, n)-u_0(x, n+1)))}{u_0(x, n)-1/u_0(x, n)} \tag{2.49}$$

and

$$\omega(x, n)=\frac{(v_0(x, n+1)\, \Delta(n+1)+d(n+1)(v_0(x, n)-v_0(x, n+1)))}{v_0(x, n)-1/v_0(x, n)}. \tag{2.50}$$

It follows from (2.35), Lemma 2.4, and the above two equations that for all n sufficiently large

$$|u_2(x, n)|>d>1$$

and

$$|v_2(x, n)|<\frac{1}{d}.$$

This and $d(n)\to 1$ imply that there is a constant c such that (2.5a) and (2.5b) hold with $s(j)=d(j+1)$. We now show the first part of (2.38). To this end substitute (1.13) into (2.32) to get (we suppress the x dependence)

$$\xi(n)=d(n+1)\,u_0(n+1)\,u_0(n)\left(\frac{\gamma(n+1)}{1-\gamma(n+1)}+1\right)\left(\frac{\gamma(n)}{1-\gamma(n)}+1\right)$$
$$-q(n)\,u_0(n)\left(\frac{\gamma(n)}{1-\gamma(n)}+1\right)+1. \tag{2.51}$$

Now subsitute (1.5) into the above equation and use the fact that $q(n)=u_0(n)+1/u_0(n)$ to find

$$\xi(n)=d(n+1)\,u_0(n+1)\,u_0(n)\frac{\gamma(n+1)\,\gamma(n)}{(1-\gamma(n+1))(1-\gamma(n))}$$
$$+d(n+1)\,u_0(n+1)\,u_0(n)\frac{\gamma(n+1)}{1-\gamma(n+1)}$$
$$+d(n+1)\,u_0(n+1)\,u_0(n)\frac{\gamma(n)}{1-\gamma(n)}$$
$$-u_0(n)\left(\frac{u_0(n)-d(n+1)\,u_0(n+1)}{u_0(n)-1/u_0(n)}\right)\frac{(u_0(n)-1/u_0(n))}{(1-\gamma(n))}(1-\gamma(n))$$
$$-(u_0(n)+1/u_0(n))\frac{\gamma(n)}{1-\gamma(n)}u_0(n). \tag{2.52}$$

Combining the third term on the RHS of the above equation with the fifth term yields

$$\xi(n)=d(n+1)\,u_0(n+1)\,u_0(n)\frac{\gamma(n+1)\,\gamma(n)}{(1-\gamma(n+1)(1-\gamma(n))}$$
$$-u_0(n)\frac{\gamma(n)^2}{1-\gamma(n)}(u_0(n)-1/u_0(n))$$
$$-\frac{\gamma(n)}{1-\gamma(n)}+d(n+1)\,u_0(n+1)\,u_0(n)\frac{\gamma(n+1)}{1-\gamma(n+1)}$$
$$-u_0(n)\frac{\gamma(n)}{1-\gamma(n)}(1-\gamma(n))(u_0(n)-1/u_0(n)).$$

Now combine the second, third, and fifth terms to get

$$\xi(n)=d(n+1)\,u_2(n+1)\,u_2(n)\,\gamma(n+1)\,\gamma(n)$$
$$+d(n+1)\,u_0(n+1)\,u_0(n)\frac{\gamma(n+1)}{1-\gamma(n+1)}-u_0^2(n)\frac{\gamma(n)}{1-\gamma(n)}.$$

This can be rewritten as

$$\begin{aligned}\xi(n) = {} & d(n+1)\, u_2(n+1)\, u_2(n)\, \gamma(n+1)\, \gamma(n) \\ & + d(n+1)\, u_2(n+1)\, u_2(n)(\gamma(n+1) - \gamma(n)) \\ & - u_2(n)(u_0(n) - 1/u_0(n))\, \gamma^2(n). \end{aligned} \tag{2.53}$$

It is easy to see from (2.49) and Lemma 2.4 that the first part of (2.35) implies the summability of the first and third terms on the right hand side of (2.53). If we difference (2.49) and use (2.40) and (2.42) it is elementary but tedious to show that (2.35) implies the summability of the second term on the right hand side of (2.53). This shows that $\xi(n)$ is summable. If we apply the same procedure to (2.33) we find

$$\begin{aligned}\zeta(n) = {} & d(n+1)\, v_2(n+1)\, v_2(n)\, \omega(n+1)\, \omega(n) \\ & + d(n+1)\, v_2(n+1)\, v_2(n)(\omega(n+1) - \omega(n)) \\ & - v_2(n)(v_0(n) - 1/v_0(n))\, \omega(n)^2, \end{aligned} \tag{2.54}$$

and it is not difficult to deduce that (2.35) implies that summability of (2.54).

Remark. Note that if N_1 is infinite (2.34) and (2.35) imply via (2.49) and (2.50) that there exists an N_0 such that (2.30) and (2.31) are satisfied for $n \geqslant N_0$.

In order to examine solutions in the oscillatory region $[-2, 2]$, we must put stronger conditions on the coefficients in the recurrence formula. We will only consider the case when N_1 is infinite as the case for N_1 finite follows as before.

THEOREM 2.5. *Suppose $d(n) \neq 0$, $n \geqslant N$,*

$$\sum_N^\infty (|\Delta(n)| + |r(x, n)|) < \infty, \tag{2.55}$$

and there exists a compact set C_1 containing an open set U such that $\{2, -2\} \subset U$. Suppose $q(x, n) \notin C_1$ for $n \geqslant N$. Finally suppose (2.30) and (2.31) hold for $n \geqslant N$. Then

$$\sum_N^\infty |\xi(n)| < \infty \qquad \textit{and} \qquad \sum_N^\infty |\zeta(n)| < \infty, \tag{2.56}$$

and there exists two solutions $y_\pm$ of (2.1) such that (2.32) holds for all n greater than N.

Proof. The hypothesis on $q(x, n)$ imply that there is a $d>0$ such that $|u_0(x, n)+(-1)^i|>d$ and $|v_0(x, n)+(-1)^i|>d$, $i=0, 1$. Thus $u_2(x, n)$ and $v_2(x, n)$ are bounded away from zero. They are also bounded away from infinity by (2.30) and (2.31). Equations (2.49) and (2.53) show that the summability of r and Δ imply the summability of ξ. We must show that

$$G(m, n)=\sum_{k=n}^{m} \prod_{j=n+1}^{k} (d(j+1)\, u_2(j+1)\, u_2(j))^{-1} \tag{2.57}$$

is bounded for all $m \geqslant n \geqslant N$. Since $d(n) \to 1$ by (2.49) we can find an N_0 such that for $n \geqslant N_0$, $d(n+1)\, u_2(n+1)\, u_2(n)$ is bounded strictly away from 1. Thus for $n \geqslant N_0$

$$G(m, n)=\sum_{k=n}^{m} \frac{1-t(k+1)}{1-t(k+1)} \prod_{j=n+1}^{k} t(j), \tag{2.58}$$

where

$$t(j)=(d(j+1)\, u_2(j+1)\, u_2(j))^{-1}. \tag{2.59}$$

Summing the LHS of (2.58) by parts gives

$$\begin{aligned} G(m, n)=&\frac{1}{1-t(m+1)} \sum_{k=n}^{m} (1-t(k+1)) \prod_{j=n+1}^{k} t(j) \\ &-\sum_{k=n}^{m} \left[\sum_{l=n}^{k-1} (1-t(l+1)) \prod_{j=n+1}^{l} t(j)\right] \\ &\times\left(\frac{1}{1-t(k+1)}-\frac{1}{1-t(k)}\right). \end{aligned}$$

If we sum the first term and the term in the brackets in the second term on the RHS of the above equation we find

$$\begin{aligned} G(m, n)=&\frac{1}{1-t(m+1)}\left(1-\prod_{j=n+1}^{m+1} t(j)\right) \\ &-\sum_{k=n+1}^{m}\left(1-\prod_{j=n+1}^{k} t(j)\right)\left(\frac{1}{1-t(k+1)}-\frac{1}{1-t(k)}\right). \end{aligned} \tag{2.60}$$

This in turn can be rewritten as

$$\begin{aligned} G(m, n)=&\frac{1}{1-t(n+1)}-\frac{\prod_{j=n+1}^{m+1} t(j)}{1-t(m+1)} \\ &+\sum_{k=n+1}^{m} \prod_{j=n+1}^{k} t(j)\left(\frac{1}{1-t(k+1)}-\frac{1}{1-t(k)}\right). \end{aligned} \tag{2.61}$$

Since $|u_0(x, n)| \geqslant 1$ implies (2.59),

$$|t(j)| \leqslant \frac{|1-\gamma(x, j+1)|\,|1-\gamma(x, j)|}{|d(j+1)|}. \tag{2.62}$$

The summability of $\gamma(x, j)$ and $(1-d(j))$, guaranteed by (2.49) and (2.55), plus the fact that $d(n)$ is bounded away from zero and $\gamma(x, j)$ is bounded away from infinity imply there exists a constant c_2 such that

$$\prod_{j=N}^{\infty} |t(j)| < c_2. \tag{2.63}$$

Thus

$$\begin{aligned} |G(m, n)| \leqslant{} & \left|\frac{1}{1-t(n+1)}\right| + \frac{c_2}{|1-t(m+1)|} \\ & + c_2 \sum_{k=n}^{m} \left|\frac{1}{1-t(k+1)} - \frac{1}{1-t(k)}\right|, \qquad m \geqslant n \geqslant N_0. \end{aligned} \tag{2.64}$$

The boundedness of $|G(m, n)|$ now follows from (2.59), (2.55), and Lemma 2.4 since $t(k)$ is strictly bounded away from one. If we write

$$G^1(m, n) = \sum_{k=m+1}^{n} \prod_{j=k}^{n} d(j+1)\, v_0(j+1)\, v_0(j) \tag{2.65}$$

then analogous considerations to those given above show that it is bounded for $n \geqslant m \geqslant N$. Thus the result is proved.

LEMMA 2.6. *Suppose in* (1.1) $d(n) = a_n/a_{n-1}$ *and* $q(x, n) = (x - b_n)/a_n)$ *where* $a_n, b_n \in \mathbb{R}$ *and* $a_n > 0$ *for all* $n \geqslant N$. *Then* (2.30) *and* (2.31) *are fulfilled for* $x \in \mathbb{C}\setminus\mathbb{R}$ *and* $n \geqslant N$.

Proof. From (1.6) and (1.7) we see that for $x \in \mathbb{C}\setminus\mathbb{R}$, $v_0(x, n) - 1/v_0(x, n)$ and $u_0(x, n) - 1/u_0(x, n)$ are not equal to zero for all $n \geqslant N$. If we look for equality in (2.30) and (2.31) we find that

$$u_0(x, n) - \frac{a_{n+1}}{a_n} u_0(x, n+1) = \left(u_0(x, n) - \frac{1}{u_0(x, n)}\right) \tag{2.66}$$

and

$$v_0(x, n) - \frac{a_{n+1}}{a_n} v_0(x, n+1) = \left(v_0(x, n) - \frac{1}{v_0(x, n)}\right). \tag{2.67}$$

Now substitute (1.6) in (2.66) and (1.7) into (2.67) and use the substitution for $q(x, n)$ indicated in the hypothesis of the lemma to find respectively

$$\sqrt{\left(\frac{x-b_n}{2a_n}\right)^2 - 1} + \frac{a_{n+1}}{a_n}\sqrt{\left(\frac{x-b_{n+1}}{2a_{n+1}}\right)^2 - 1} = \pm\frac{b_{n+1}-b_n}{2a_n}, \qquad (2.68)$$

where the plus sign is associated with (2.66) and the minus sign with (2.67). The result will follow if we show that for x complex the LHS of (2.68) has an imaginary part since the RHS by hypothesis is real. Since the imaginary part of the LHS of (2.68) is harmonic for $\operatorname{Im} x > 0$, examining the values of this function for $\operatorname{Im} x = 0$ and for $|x| \to \infty$ and using the determination of the square root given by (1.8), we conclude from the minimum principle that it cannot be equal to zero for $\operatorname{Im} x > 0$. Analogous considerations show that the imaginary part of the LHS of (2.68) does not vanish for $\operatorname{Im} x < 0$.

3. Recurrence Formulas with Regularly Varying Coefficients

We now apply the previous results to some specific examples. In (1.1) let

$$d(n+1) = \frac{a_{n+1}}{a_n} \qquad \text{and} \qquad q(x, n) = \frac{x-b_n}{a_n}, \qquad n > 0, \qquad (3.1)$$

where the a_n's and b_n's are regularly varying at infinity, i.e., there exists an increasing positive sequence $\{\lambda_n, n = 0, 1, ...\}$ such that

$$\lim_{n\to\infty} \frac{a_n}{\lambda_n} = a > 0, \qquad \lim_{n\to\infty} \frac{b_n}{\lambda_n} = b \in \mathbb{R}, \qquad (3.2)$$

with

$$\lim_{n\to\infty} n\left(\frac{\lambda_{n+1}}{\lambda_n} - 1\right) = \alpha \geqslant 0. \qquad (3.3)$$

α is called the index of regular variation. We assume that $a_{n+1} > 0$, $b_n \in \mathbb{R}$, $n = 0, 1, 2, ...,$ and

$$\lim_{n\to\infty} n\frac{(a_{n+1}-a_n)}{\lambda_n} = a\alpha, \qquad \lim_{n\to\infty} n\frac{(b_{n+1}-b_n)}{\lambda_n} = b\alpha. \qquad (3.4)$$

From the theory of regularly varying sequences (Bojanic and Seneta [3]), (3.2) and (3.3) imply that there exist functions $l(x)$, $l_1(x)$, and $l_2(x)$ defined by

$$l(x) = \frac{\lambda_{[x]}}{[x]^\alpha}, \qquad l_1(x) = \frac{a_{[x]}}{a[x]^\alpha}, \qquad l_2(x) = \frac{b_{[x]}}{b[x]^\alpha}, \qquad b \neq 0 \qquad (3.5)$$

such that

$$\lim_{x\to\infty}\frac{l(tx)}{l(x)}=\lim_{x\to\infty}\frac{l_1(tx)}{l_1(x)}=\lim_{x\to\infty}\frac{l_2(tx)}{l_2(x)}=1 \tag{3.6}$$

for every $t>0$. Here $[x]$ means the integer part of x. Functions satisfying (3.6) are said to be slowly varying. Therefore a_n and b_n have the representation

$$a_n=an^{\alpha}l_1(n) \qquad \text{and} \qquad b_n=bn^{\alpha}l_2(n). \tag{3.7}$$

Equation (3.2) implies that

$$\lim_{x\to\infty}\frac{l_1(x)}{l(x)}=\lim_{x\to\infty}\frac{l_2(x)}{l(x)}=1. \tag{3.8}$$

Equations (3.3), (3.4), and (3.5) imply that

$$\lim_{n\to\infty} n\left(\frac{l(n+1)}{l(n)}-1\right)=\lim_{n\to\infty} n\left(\frac{l_1(n+1)}{l_1(n)}-1\right)$$
$$=\lim_{n\to\infty} n\left(\frac{l_2(n+1)}{l_2(n)}-1\right)=0. \tag{3.9}$$

It follows from the theory of regularly varying sequences [3, Theorem 4] that

$$\lim_{n\to\infty}\frac{a_{[nt]}}{a_n}=t^{\alpha} \qquad \text{and} \qquad \lim_{n\to\infty}\frac{b_{[nt]}}{b_n}=t^{\alpha} \tag{3.10}$$

for every $t>0$ and (3.10) coupled with (3.4) gives

$$\lim_{n\to\infty} n\left(\frac{a_{[nt]+1}-a_{[nt]}}{\lambda_n}\right)=a\alpha t^{\alpha-1}. \tag{3.11}$$

and

$$\lim_{n\to\infty} n\left(\frac{b_{[nt]+1}-b_{[nt]}}{\lambda_n}\right)=b\alpha t^{\alpha-1}, \tag{3.12}$$

for $t>0$.

Lemma 3.1. *Let a_n and b_n be given by (3.7) then*

$$\frac{1}{\lambda_n^2}\sum_{i=1}^{n}(a_{i+1}-a_i)^2\leqslant c\frac{1}{n^{\rho}} \qquad \textit{and} \qquad \frac{1}{\lambda_n^2}\sum_{i=1}^{n}(b_i-b_{i-1})^2\leqslant c\frac{1}{n^{\rho}}, \tag{3.13}$$

with c a constant and

$$\rho=\begin{cases}1 & & \alpha>\frac{1}{2}\\ 1-\varepsilon & \varepsilon>0 & \alpha=\frac{1}{2}\\ 2\alpha-\varepsilon & \varepsilon>0 & \alpha<\frac{1}{2}.\end{cases} \tag{3.14}$$

Furthermore

$$\frac{1}{\lambda_n}\sum_{i=1}^{n}|a_{i+1}-2a_i+a_{i-1}|=o(1) \qquad \alpha>0 \tag{3.15}$$

and

$$\frac{1}{\lambda_n}\sum_{i=1}^{n}|b_{i+1}-2b_i+b_{i-1}|=o(1) \qquad \alpha>0. \tag{3.16}$$

If

$$\frac{[a_{i+1}-2a_i+a_{i-1}]}{a_i}=O(i^{-1-\beta})=\frac{[b_{i+1}-2b_i+b_{i-1}]}{b_i} \qquad 0<\beta\leqslant 1 \tag{3.17}$$

for i large enough, then the $o(1)$ *in* (3.15) *and* (3.16) *may be replaced by*

$$O(1/n^{\sigma}), \qquad \sigma=\begin{cases}1 & & \alpha>1\\ 1-\varepsilon & \varepsilon>0 & \alpha=1\\ \alpha-\varepsilon & \varepsilon>0 & \alpha<1.\end{cases} \tag{3.18}$$

Proof. If we write

$$\begin{aligned} a_{i+1}-a_i &= a_i\left(\left(\frac{i+1}{i}\right)^{\alpha}\frac{l_1(i+1)}{l_1(i)}-1\right)\\ &= a_i\left(\frac{\alpha}{i}+\frac{l_1(i+1)}{l_1(i)}-1+O(1/i^2)\right), \end{aligned} \tag{3.19}$$

where (3.9) has been used. This implies that $(a_{i+1}-a_i)$ is regularly varying with index $\alpha-1$, hence $(a_{i+1}-a_i)^2$ is regularly varying with index $2(\alpha-1)$. The first part of (3.13) now follows from Karamata's theorem for regularly varying sequences [3, Theorem 6] which says that if $c(k)$ is regularly varying with index α then

$$\lim_{n\to\infty}\frac{1}{n^{1+\beta}c(n)}\sum_{k=1}^{n}k^{\beta}c(k)=\frac{1}{1+\beta+\alpha}, \tag{3.20}$$

for $\beta > -\alpha - 1$. If $\alpha \leqslant \frac{1}{2}$, the result follows from Potter's bound [2, p. 25] which says for every $\varepsilon > 0$ and every slowly varying function $l(x)$ there exist constants c_1 and c_2 such that $c_1 x^{-\varepsilon} < l(x) < c_2 x^{\varepsilon}$ for x sufficiently large. To show the second part of (3.13) we write

$$b_{i+1} - b_i = b_i \left(\frac{\alpha}{i} + \frac{l_2(i+1)}{l_2(i)} - 1 + O(1/i^2) \right), \qquad b \neq 0.$$

Here i is assumed large enough so that $l_2(i) \neq 0$ (see (3.9)). Following the reasoning above leads to the second part of (3.13).

To show (3.15), difference (3.19) to find

$$|a_{i+1} - 2a_i + a_{i-1}| = a_{i-1} O \left[\left(\frac{\alpha}{i} + \left| \frac{l_1(i+1)}{l_1(i-1)} - 1 \right| \right)^2 + \frac{\alpha}{i^2} + \left| \frac{l_1(i+1)}{l_1(i)} - \frac{l_1(i)}{l_1(i-1)} \right| \right].$$

That (3.15) holds now follows from the Toeplitz–Silverman theorem [6, p. 43] and (3.20) with $c(k) = \lambda_k$ and $\beta = -1$. If the first part of (3.17) is satisfied then Karamata's theorem and Potter's bound show that $o(1)$ in (3.15) may be replaced by $O(1/n^{\sigma})$, where σ is given in (3.18). Analogous arguments applied to $b_{i+1} - 2b_i + b_{i-1}$ yield the result.

Set

$$[A, B] = \text{convex hull}(\{0\}, [b - 2a, b + 2a]). \tag{3.21}$$

LEMMA 3.2. *Suppose $\{a_i\}$ and $\{b_i\}$ are regularly varying sequences with index of regular variation α, $a_i \to \infty$, and $|b_i| \to \infty$ or $b = 0$. If $\alpha = 0$ suppose that*

$$\limsup \left\{ \max_{1 \leqslant k \leqslant n} \frac{a_k}{a_n} \right\} = 1 = \limsup \left\{ \max_{0 \leqslant k \leqslant n} \frac{b_k}{b_n} \right\}, \qquad b \neq 0. \tag{3.22}$$

Then for $y \notin [A, B]$ there exist a $d > 1$, an N_0, and a fixed, finite N_1 such that for $n \geqslant N_0$

$$|u_0(\lambda_n y, i)| > d \qquad i = 1, 2, ..., n + N_1, \tag{3.23a}$$

$$|v_0(\lambda_n y, i)| < 1/d \qquad i = 1, 2, ..., n + N_1, \tag{3.23b}$$

$$|\gamma(\lambda_n y, i)| < 1/d \qquad i = 1, 2, ..., n + N_1, \tag{3.23c}$$

and

$$|\omega(\lambda_n y, i)| < 1/d \qquad i = 1, 2, ..., n + N_1, \tag{3.23d}$$

where u_0, v_0, γ, and ω are defined in Eqs. (1.6), (1.7), (2.47), *and* (2.48), *respectively.*

Proof. Since a_n and b_n are regularly varying sequences one finds [3, Theorem 5]

$$\max_{1\leqslant k\leqslant n}\left\{\frac{k^{-\sigma}a_k}{n^{-\sigma}a_n}\right\}\sim 1\sim \max_{1\leqslant k\leqslant n}\left\{\frac{k^{-\sigma}b_k}{n^{-\sigma}b_n}\right\},\qquad \sigma<\alpha, \tag{3.24}$$

and

$$\min_{1\leqslant k\leqslant n}\left\{\frac{k^{-\tau}a_k}{n^{-\tau}a_n}\right\}\sim 1\sim \min_{1\leqslant k\leqslant n}\left\{\frac{k^{-\tau}b_k}{n^{-\tau}b_n}\right\},\qquad \tau>\alpha, \tag{3.25}$$

where we have assumed $b\neq 0$ in the second parts of (3.24) and (3.25). Suppose for convenience that $b>0$, then for y real and greater than $b+2a$

$$\liminf \min_{1\leqslant k\leqslant n}\left(\frac{y-b_k/\lambda_n}{2(a_k/\lambda_n)}\right)\geqslant \inf_{t\in[0,1]} t^{-\mu}\left(\frac{y-b}{2a}\right)=\frac{y-b}{2a}>1,\qquad \mu\geqslant 0,$$

where (3.24) with $\alpha>0$ or (3.22) with $\alpha=0$ has been used. If y is less than $b+2a$ and $y\notin[A,B]$ it must be negative and less than $b-2a$. Using (3.24) and (3.25) for $\alpha>0$ or (3.22) and (3.25) for $\alpha=0$ we find that

$$\limsup \max_{1\leqslant k\leqslant n}\left(\frac{y-b_k/\lambda_n}{2(a_k/\lambda_n)}\right)\leqslant\frac{y-b}{2a}<-1. \tag{3.26}$$

For y complex we examine the imaginary part of $\lambda_n y-b_k/a_k$. For $\operatorname{Im} y>0$ (3.22), for $\alpha=0$, or (3.24), for $\alpha>0$, implies that $\liminf \min_{1\leqslant k\leqslant n} \operatorname{Im} y/(a_k/\lambda_n)=\operatorname{Im} y$, while for $\operatorname{Im} y<0$ the same equations say that $\limsup \max_{1\leqslant k\leqslant n} \operatorname{Im} y/(a_k/\lambda_n)=\operatorname{Im} y$. Since $|z+\sqrt{z^2-4}|>1$ for $z\notin[-2,2]$ the above arguments show that for $y\notin[A,B]$ there exist $d>1$, an N_0, and an N_1 such that for all $n\geqslant N_0$

$$|u_0(\lambda_n y, i)|>d\qquad i=1,2,...,n+N_1. \tag{3.27}$$

Similar manipulations can be used for the cases $b=0$ and $b<0$ to arrive at (3.27). Since $v_0(\lambda_n y, i)=1/u_0(\lambda_n y, i)$, (3.23b) follows from (3.23a). To prove (3.23c) write

$$\begin{aligned}\gamma(\lambda_n y, i)&=\frac{u_0(\lambda_n y, i)-(a_{i+1}/a_i)u_0(\lambda_n y, i+1)}{u_0(\lambda_n y, i)-1/u_0(\lambda_n y, i)}\\&=\left(\frac{b_{i+1}-b_i}{2a_i}+\sqrt{\left(\frac{\lambda_n y-b_i}{2a_i}\right)^2-1}\right.\\&\qquad\left.-\frac{a_{i+1}}{a_i}\sqrt{\left(\frac{\lambda_n y-b^{i+1}}{2a_{i+1}}\right)^2-1}\right)\Bigg/\left(2\sqrt{\left(\frac{\lambda_n y-b_i}{2a_i}\right)^2-1}\right).\end{aligned} \tag{3.28}$$

From (3.22) and (3.25) for $\alpha=0$ or (3.24) and (3.25) for $\alpha>0$ we find that for sufficiently large n and $y\notin[A,B]$ there exists a constant $c>0$ such that

$$\left|\frac{2a_i}{\lambda_n}\left(u_0(\lambda_n y, i)-\frac{1}{u_0(\lambda_n y, i)}\right)\right|=2\left|\sqrt{\left(y-\frac{b_i}{\lambda_n}\right)^2-\frac{4a_i^2}{\lambda_n^2}}\right|>c. \tag{3.29}$$

Clearing the radicals in the numerator of (3.28), then using (3.29), yields

$$|\gamma(\lambda_n y, i)|\leqslant\frac{1}{c}\left|\frac{b_{i+1}-b_i}{\lambda_n}\right|+\frac{1}{c^2}\left|\frac{b_{i+1}-b_i}{\lambda_n}2y+\frac{b_i^2-b_{i+1}^2}{\lambda_n^2}+\frac{a_i^2-a_{i+1}^2}{\lambda_n^2}\right|. \tag{3.30}$$

Equation (3.4) and the condition $\lambda_n\to\infty$ now give (3.23c). Equation (3.23d) follows from

$$|\omega(\lambda_n y, i)|\leqslant\frac{1}{c}\left|\frac{b_{i+1}-b_i}{\lambda_n}\right|+\frac{1}{c^2}\left|\frac{b_{i+1}-b_i}{\lambda_n}2y+\frac{b_i^2-b_{i+1}^2}{\lambda_n^2}+\frac{a_i^2-a_{i+1}^2}{\lambda_n^2}\right|,$$

which is arrived at starting from (2.48) and using manipulations similar to those used leading to (3.30).

LEMMA 3.3. *Suppose* (3.2), (3.3), *and* (3.4) *hold with* $a(i)\to\infty$ *and* $|b(i)|\to\infty$ *or* $b=0$. *Suppose* $y\notin[A,B]$, N_0, N_1 *are as is Lemma* 3.2 *and* $\xi(\lambda_n y, i)$ *and* $\zeta(\lambda_n y, i)$ *are given by* (2.33a) *and* (2.33b) *respectively. If* $\alpha>0$ *then*

$$\sum_{j=1}^{n}\left|\frac{\xi(\lambda_n y, j)}{(a_{j+1}/a_j)\,u_2(\lambda_n y, j+1)\,u_2(\lambda_n y, j)}\right|=o(1),\qquad n\geqslant N_0. \tag{3.31}$$

and

$$\sum_{j=1}^{n+N_1}|\zeta(\lambda_n y, j)|=o(1),\qquad n\geqslant N_0. \tag{3.32}$$

If $\alpha=0$ *suppose* (3.22) *holds and*

$$\frac{1}{\lambda_n}\sum_{i=2}^{n}|a_{i+1}-2a_i+a_{i-1}|=o(1)=\frac{1}{\lambda_n}\sum_{i=1}^{n}|b_{i+1}-2b_i+b_{i-1}|, \tag{3.33}$$

then (3.31) *and* (3.32) *are still valid.*

Proof. From (2.53) and Lemma 3.2 we find that for $n \geqslant N_0$

$$\left|\frac{\xi(\lambda_n y, j)}{(a(j+1)/a(j))\, u_2(\lambda_n y, j+1)\, u_2(\lambda_n y, j)}\right| \leqslant |\gamma(\lambda_n y, j+1)\, \gamma(\lambda_n y, j)| + |\gamma(\lambda_n y, j+1) - \gamma(\lambda_n y, j)| + \left|\frac{u_0(\lambda_n y, j) - 1/u_0(\lambda_n y, j)}{(a(j+1)/a(j))\, u_2(\lambda_n y, j+1)}\right| |\gamma(\lambda_n y, j)|^2.$$

Since $|u_0(\lambda_n y, j)/u_0(\lambda_n y, j+1)|$ is finite and $|u_0(\lambda_n y, j)| \geqslant 1$ for all n and j, Lemma 3.2 implies that

$$\left|\frac{u_0(\lambda_n y, j) - 1/u_0(\lambda_n y, j)}{(a(j+1)/a(j))\, u_2(\lambda_n y, j+1)}\right| \leqslant 2C\left(1 + \frac{1}{d}\right).$$

From (2.47) and (2.48) we find

$$\sum_{i=1}^{n} |\gamma(\lambda_n y, i+1) - \gamma(\lambda_n y, i)| \leqslant \frac{C}{\lambda_n} \sum_{i=1}^{n} [|a_{i+2} - 2a_{i+1} + a_i| + |b_{i+2} - 2b_{i+1} + b_i|] + \frac{C_1}{\lambda_n^2} \sum_{i=1}^{n} [|a_{i+1} - a_i|^2 + |b_{i+1} - b_i|^2]$$

and

$$\sum_{i=1}^{n+N_1} |\omega(\lambda_n y, i+1) - \omega(\lambda_n y, i)| \leqslant \frac{C}{\lambda_n} \sum_{i=1}^{n+N_1} [|a_{i+2} - 2a_{i+1} + a_i| + |b_{i+2} - 2b_{i+1} + b_i|] + \frac{C_1}{\lambda_n^2} \sum_{i=1}^{n+N_1} [|a_{i+1} - a_i|^2 + |b_{i+1} - b_i|^2].$$

If $\alpha > 0$ the result now follows from Lemma 3.1 while if $\alpha = 0$ the result follows from (3.9) and (3.33).

THEOREM 3.4. *Suppose* $y \notin [A, B]$ *and* (3.2), (3.3), *and* (3.4) *hold with* $a_i \to \infty$ *and* $|b_i| \to \infty$ *or* $b = 0$. *If* $\alpha > 0$ *then there exist* $N_0 > 0$, N_1, *and solu-*

tions $y_+(\lambda_n y, m)$ *and* $y_-(\lambda_n y, m)$ *of* (1.1) *with the substitutions given by* (3.1) *such that for* $n \geqslant N_0$

$$\left| y_+(\lambda_n y, m) \prod_{i=1}^{m} u_2(\lambda_n y, i)^{-1} - 1 \right| \leqslant \exp\left\{ C \sum_{j=1}^{m} \left| \frac{\zeta(\lambda_n y, j)}{(a_{j+1}/a_j)\, u_2(\lambda_n y, j+1)\, u_2(\lambda_n y, j)} \right| \right\} - 1, \quad m = 1, 2, ..., n + N_1, \tag{3.34}$$

and

$$\left| y_-(\lambda_n y, m) \prod_{i=1}^{m} v_2(\lambda_n y, i)^{-1} - 1 \right| \leqslant \exp\left\{ C \sum_{j=1}^{m} |\zeta(\lambda_n y, j)| \right\} - 1, \qquad m = 1, 2, ..., n + N_1. \tag{3.35}$$

The RHS *of the above inequalities tend to zero as n tends to infinity.*

If $\alpha = 0$ *and* (3.22) *holds then* (3.34) *and* (3.35) *are still valid; furthermore if* (3.33) *is also true then the* RHS *in the above inequalities tend to zero as n tends to infinity.*

Proof. In order to prove (3.34) we must show that for $y \notin [A, B]$ and $n \geqslant N_0$ there exists a C such that

$$\sum_{k=i}^{m-1} \sum_{j=i+1}^{k} \frac{1}{(a_{j+1}/a_j)\, |u_2(\lambda_n y, j+1)\, u_2(\lambda_n y, j)|} < C, \quad n + N_1 > m > i > 0. \tag{3.36}$$

and this follows from Lemma 3.2. (Note that j starts from $i+1$ instead of i because of the denominator in the LHS of (3.34).) That the error term is $o(1)$ in (3.34) when $\alpha > 0$ or if $\alpha = 0$ when (3.22) and (3.33) hold is a consequence of Lemma 3.3. In order to prove (3.35) we must show that for $y \notin [A, B]$ and $n \geqslant N_0$ there exists a C such that

$$\sum_{k=m+1}^{i} \sum_{j=k}^{i-1} \frac{a_{j+1}}{a_j} |v_2(\lambda_n y, j)\, v_2(\lambda_n y, j)| < C, \qquad n + N_1 \geqslant i > m > 0. \tag{3.37}$$

which again follows from Lemma 3.2. Lemma 3.3 can now be invoked to determine the decay of the error terms.

THEOREM 3.5. *Suppose* $y \notin [A, B]$, *and* (3.2), (3.3), *and* (3.4) *hold with* $a_i \to \infty$ *and* $|b_i| \to \infty$ *or* $b = 0$. *Let* $p(x, n)$ *be the solution of* (1.1), *with* $d(n)$

and q given by (3.1), *satisfying the initial conditions* $p(x,-1)=0$, $p(x,0)=1$. *If* $\alpha>0$, *or for* $\alpha=0$ *if* (3.22) *and* (3.33) *hold, then*

$$\left|p(\lambda_n y, n)\prod_{i=1}^{n} u_2(\lambda_n y, i)^{-1}-1\right|=o(1). \tag{3.38}$$

Proof. We begin by examining the Wronskian, W, of y_+ and y_- given by $W(y_+, y_-)(i) = a_{i+1}(y_+(\lambda_n y, i+1)\, y_-(\lambda_n y, i) - y_+(\lambda_n y, i)\, y_-(\lambda_n y, i+1))$. $W(y_+, y_-)$ is independent of i and may be evaluated by setting i equal to one, which yields

$$\begin{aligned} W(y_+, y_-)(1) = a_2 u_2(\lambda_n y, 1)\, v_2(\lambda_n y, 1)(u_2(\lambda_n y, 2)(1+\phi(\lambda_n y, 2)) \\ - v_2(\lambda_n y, 2)(1+\hat{\phi}(\lambda_n y, 2)). \end{aligned}$$

Equations (2.7), (2.20), and (3.1) with $f(i)=u_2(\lambda_n y, i)$ and $g(i)=v_2(\lambda_n y, i)$ have been used to arrive at the above equation. If we now use (1.6), (1.7), (1.13), (1.14) and Lemma 3.2 we find

$$W(y_+, y_-)=a_2\lambda_n\left(\left(\frac{u_0(\lambda_n y, 2)}{\lambda_n}-\frac{v_0(\lambda_n y, 2)}{\lambda_n}\right)(1+o(1))\right). \tag{3.39}$$

Since $u_0(\lambda_n y, 2)/\lambda_n \to y/a_2$ and $v_0(\lambda_n y, 2)/\lambda_n \to 0$, (3.39) implies that $y_+(\lambda_n y, i)$ and $y_-(\lambda_n y, i)$ are linearly independent for n sufficiently large and $i=1, 2, ..., n+N_1$. Therefore

$$p(\lambda_n y, i)=Cy_+(\lambda_n y, i)+Dy_-(\lambda_n y, i), \tag{3.40}$$

where $D=-W[p, y_+]/W[y_+, y_-]$, and $C=W[p, y_-]/W[y_+, y_-]$. If we define $y_+(x,0)=((x-b_1)/a_1)\, y_+(x,1)-(a_2/a_1)\, y_+(x,2)$ and use (2.7), (2.14) (all with $N=1$), and (2.1) then $y_+(x,0)=1$. D can now be evaluated using the above equation, the initial conditions satisfied by $p(x,n)$ and (3.39) which give $D=(a_1[p(\lambda_n y, 1)-y_+(\lambda_n, 1)]/\lambda_n y)(1+o(1))=o(1)$, where $\lim_{n\to\infty} p(\lambda_n y, 1)/\lambda_n = a_1 = \lim_{n\to\infty} y_+(\lambda_n y, 1)/\lambda_n$ has been used. Thus (3.40) becomes

$$\frac{p(\lambda_n y, i)}{\prod_{i=1}^{n} u_2(\lambda_n y, i)}=\frac{Cy_+(\lambda_n y, i)}{\prod_{i=1}^{n} u_2(\lambda_n y, i)}+o(1). \tag{3.41}$$

The fact that $y_-(\lambda_n y, n)/\prod_{i=1}^{n} u_2(\lambda_n y, i)\to 0$ as n tends to infinity (see Theorem 3.4) has been used to obtain the $o(1)$ term in (3.41). To evaluate C we note that

$$C=\frac{[p(\lambda_n y, 2)\, y_-(\lambda_n y, 1)-p(\lambda_n y, 1)\, y_-(\lambda_n y, 2)]}{[y_+(\lambda_n y, 2)\, y_-(\lambda_n y, 1)-y_+(\lambda_n y, 1)\, y_-(\lambda_n y, 2)]}. \tag{3.42}$$

It follows from (1.7) that $v_0(z) \sim O(1/z)$ and this combined with Theorem 3.4 shows that $\lim_{n\to\infty} p(\lambda_n y, 1)\, y_-(\lambda_n y, 2) = 0$ and $\lim_{n\to\infty}(p(\lambda_n y, 2)\, y_-(\lambda_n y, , 1) - \lambda_n y/a_2) = 0$. Using these results and (3.39) we find $C = 1 + o(1)$. Substituting this into (3.40) then using Theorem 3.4 gives the result.

COROLLARY 3.6. *Under the hypothesis of Theorem* 3.5 *it follows that*

$$\lim_{n\to\infty} \frac{p(\lambda_n y, n)}{\prod_{i=1}^{n} u_0(\lambda_n y, i)(1+\gamma(\lambda_n y, i))} = 1, \qquad y \notin [A, B], \tag{3.43}$$

where the convergence is uniform on compact subsets of $\mathbb{C}\backslash[A, B]$. *Here* $\gamma(\lambda_n y, i)$ *is given by* (2.47), *and* $\mathbb{C}$ *is the complex plane.*

Proof. From (3.38) we find that

$$\lim_{n\to\infty} \frac{p(\lambda_n y, n)}{\prod_{i=1}^{n} u_2(\lambda_n y, i)} = 1, \tag{3.44}$$

where the convergence is uniform on compact subsets of $\mathbb{C}\backslash[A, B]$. From (1.13) and (3.30) it follows that

$$u_2(\lambda_n y, i) = u_0(\lambda_n y, i)\left(1 + \gamma(\lambda_n y, i) + O\left(\left(\frac{|a_{i+1} - a_i|}{\lambda_n} + \frac{|b_{i+1} - b_i|}{\lambda_n}\right)^2\right)\right). \tag{3.45}$$

By Lemma 3.2 $|\gamma(\lambda_n y, i)| < 1$ for n sufficiently large and $i = 1, 2, ..., n$. The result now follows by extracting $1 + \gamma(\lambda_n y, i)$ from the parenthesis in (3.45) and using Lemma 3.1.

Although (3.5) gives slowly varying functions $l_1(x)$ and $l_2(x)$ such that $a_n = an^\alpha l_1(n)$ and $b_n = bn^\alpha l_2(n)$, $l_1(x)$ and $l_2(x)$ are not unique since they are determined only at the positive integers. We shall replace these by slowly varying functions

$$\begin{aligned} L_i(x) &= (l_i(n+2) - 2l_i(n+1) + l_i(n))((x-n)^3 - (x-n)^2) \\ &\quad + (l_i(n+1) - l_i(n))(x-n) + l_i(n), \qquad n \leqslant x \leqslant n+1,\ i = 1, 2 \end{aligned} \tag{3.46}$$

where for $L_1(x)$ we take $n \geqslant 1$ while for $L_2(x)$ we let $n \geqslant 0$. We note that this choice of $L_i(x)$, $i = 1, 2$, gives

(a) $L_i(n) = l_i(n)$,

(b) $L_i'(n) = l_i(n+1) - l_i(n)$, and

(c) $L_i''(x) = O[l_i(n+2) - 2l_i(n+1) + l_i(n)]$ for $n < x < n+1$.

LEMMA 3.7. *Let L_1 and L_2 be as above; then L_1 and L_2 are C^1 slowly varying functions,* $\lim_{x\to\infty} L_1(x)/l_1(x) = 1 = \lim_{x\to\infty} L_2(x)/l_2(x)$. *If*

$$R_1(x) = x^\alpha L_1(x) \qquad \alpha \geqslant 0 \tag{3.47}$$

and

$$R_2(x) = x^\alpha L_2(x) \qquad \alpha \geqslant 0, \tag{3.48}$$

where α is given by (3.4), *then*

$$a_n = aR_1(n) \tag{3.49}$$

and

$$b_b = bR_2(n). \tag{3.50}$$

If $\alpha > 0$ then

$$\frac{1}{\lambda_n^2}\int_M^n |R_i'(y)|^2\,dy = o(1) = \frac{1}{\lambda_n}\int_M^n |R_i''(y)|\,dy, \qquad i = 1, 2. \tag{3.51}$$

If $\alpha = 0$ and (3.33) *holds then* (3.51) *is still valid.*

Proof. The continuity of $L_1(x)$ and its derivative follow from (3.46) (see (b) above). From (3.2) and (3.5) (see also Bingham, Goldie, and Teugels [2, p. 6]) we find $l_1(x)/l_1(n) \to 1$ as $n \to \infty$ uniformly for $x \in [n, n+1]$. Therefore $L_1(x)/l_1(x) \to 1$ which with (3.6) implies that $L_1(xt)/L_1(x) \to 1$ as $x \to \infty$ for every $t > 1$. Hence $L_1(x)$ is slowly varying.

Now differentiate (3.46) with $i = 1$ to find $R_1'(y) = \alpha(R_1(y)/y + y^\alpha L_1'(y)$. Therefore

$$\frac{1}{\lambda_n^2}\int_M^n |R_1'(y)|^2\,dy \leqslant \frac{2}{\lambda_n^2}\left[\int_M^n \alpha^2 \left|\frac{R_1(y)}{y}\right|^2 dy + \int_M^n y^{2\alpha}\,|L_1'(y)|^2\,dy\right], \tag{3.52}$$

where for convenience we take M to be an integer. The first term in the above equation can be bounded in the following fashion:

$$\frac{1}{\lambda_n^2}\int_M^n \left|\frac{R_1(y)}{y}\right|^2 dy \leqslant \frac{1}{\lambda_n^2}\sum_{i=M}^{n-1}\frac{1}{i^2}\int_i^{i+1} |R_1(x)|^2\,dx. \tag{3.53}$$

From (3.46) and (3.47) we find that

$$(R_1(x))^2 \leqslant 3x^{2\alpha}\{C_2(i)^2((x-i)^3 - (x-i)^2)^2 + C_1(i)^2(x-i)^2 + l_1(i)^2\},$$
$$i \leqslant x \leqslant i+1,$$

with

$$C_1(i) = l_1(i+1) - l_1(i) = \frac{a_{i+1}}{(i+1)^\alpha} - \frac{a_i}{i^\alpha} \tag{3.54}$$

and

$$C_2(i) = C_1(i+1) - C_1(i). \tag{3.55}$$

Consequently (3.53) is bounded by

$$\frac{3}{\lambda_n^2} \sum_{i=M}^{n-1} \frac{(i+1)^{2\alpha}}{i^2} (C_1(i)^2 + C_2(i)^2 + l_1(i)^2) = O\left(\frac{1}{a_n^2} \sum_{i=M}^{n+1} \frac{a_i^2}{i^2}\right), \tag{3.56}$$

where the last equality follows from (3.2), (3.7), and (3.54). Hence from (3.20) we find $(1/\lambda_n^2) \int_M^n (R_1(y)/y)^2 \, dx = o(1)$. Analyzing the second term on the RHS of (3.52) yields

$$\frac{1}{\lambda_n^2} \int_M^n y^{2\alpha} \, |L_1'(y)|^2 \, dy \leqslant \frac{2}{\lambda_n^2} \sum_{i=M}^{n-1} (i+1)^{2\alpha} (C_1^2(i) + C_2^2(i)),$$

so that $(1/\lambda_n^2) \int_M^n y^{2\alpha} \, |L_1'(y)|^2 \, dy = o(1)$. This coupled with the above remarks gives the first part of (3.51). To show the second part of (3.51) write $R_1''(y) = \alpha(\alpha-1) R_1(y)/y^2 + 2\alpha y^{\alpha-1} L_1'(y) + y^\alpha L_1''(y)$. Now calculations similar to the ones that led to (3.56) give

$$\frac{1}{\lambda_n} \int_M^n |y^{\alpha-1} L_1'(y)| \, dy \leqslant \frac{1}{\lambda_n} \sum_{i=M}^{n-1} (i+1)^{\alpha-1} (|C_1(i)| + |C_2(i)|)$$

$$= O\left(\frac{1}{a_n} \sum_{i=M}^{n} \frac{a_i}{i} \left|\frac{a_{i+1}}{a_i} - 1\right|\right) = o(1), \tag{3.57}$$

where Eqs. (3.2) and (3.4) have been used to obtain the last equality. Finally we examine

$$\frac{1}{\lambda_n} \int_M^n |y^\alpha L_1''(y)| \, dy \leqslant \frac{6}{\lambda_n} \sum_{i=M}^{n-1} (i+1)^\alpha C_2(i). \tag{3.58}$$

From (3.54) and (3.7) we find

$$i^\alpha \, |C_2(i)| = \frac{i^\alpha}{a} \left| \frac{a_{i+2}}{(i+2)^\alpha} - \frac{2a_{i+1}}{(i+1)^\alpha} + \frac{a_i}{i^\alpha} \right|$$

$$\leqslant \frac{1}{a} \, |a_{i+2} - 2a_{i+1} + a_i| + \frac{C}{a} \left| \frac{a_{i+2} - a_{i+1}}{i} \right| + \frac{da_{i+1}}{ai^2}.$$

Therefore $\sum_{i=M}^{n-1} i^\alpha |C_2(i)| = o(1)$ by Lemma 3.1 and (3.20), and the second half of (3.51) follows.

If $\alpha = 0$ and (3.33) holds we find from (3.46) that $R_1'(y) = L_1'(y)$ and

$$\frac{1}{\lambda_n^2} \int_M^n |L_1'(y)|^2 \, dy \leqslant \frac{2}{\lambda_n^2} \sum_{i=M}^{n-1} \{C_1(i)^2 + C_2(i)^2\}$$
$$\leqslant O\left(\frac{1}{a_n^2} \sum_{i=M}^{n-1} a_i^2 \left|\frac{a_{i+1}}{a_i} - 1\right|^2\right),$$

which is $o(1)$ by (3.2) and (3.4). The second part of (3.46) with $\alpha = 0$ follows immediately from (3.33) since

$$\frac{1}{\lambda_n} \int_M^n |R_1''(y)| \, dy \leqslant \frac{6}{\lambda_n} \sum_{i=M}^{n-1} \{|C_2(i)|\}.$$

The proof of (3.51) for $R_2(y)$ is the same as that given above.

We are now ready to prove the main result of this section.

THEOREM 3.8. *Suppose* $y \notin [A, B]$ *and* (3.2), (3.3), *and* (3.4) *hold with* $a_i \to \infty$ *and* $|b_i| \to \infty$ *or* $b = 0$. *If* $\alpha > 0$, *or for* $\alpha = 0$ *if* (3.22) *and* (3.33) *hold, then*

$$\lim_{n \to \infty} \frac{p(\lambda_n y, n)}{\prod_{i=1}^n u_0(\lambda_n y, i)}$$
$$= \left\{\frac{(x-b)^2 - 4a^2}{x^2}\right\}^{-1/4} \exp\left\{\frac{b}{2} \int_0^1 \frac{ds}{\sqrt{(x-bs)^2 - 4a^2b^2}}\right\} \tag{3.59}$$

uniformly on compact subsets of $\mathbb{C} \backslash [A, B]$.

Remark. This result for $\alpha > 0$ was first proved by Van Assche and Geronimo [21].

Proof. Lemma 3.2 implies that for any compact set $K \subset \mathbb{C} \backslash [A, B]$ there exists an N_0 such that for $n \geqslant N_0$, $|\gamma(\lambda_n y, i)| < 1$, $i = 1, 2, ..., n$. Therefore

$$\ln(1 + \gamma(\lambda_n y, i)) = \gamma(\lambda_n y, i) \int_0^1 \frac{d\beta}{1 + \beta\gamma(\lambda_n y, i)}$$
$$= \gamma(\lambda_n y, i) - \gamma(\lambda_n y, i)^2 \int_0^1 \frac{\beta \, d\beta}{1 + \beta\gamma(\lambda_n y, i)}. \tag{3.60}$$

From (3.9) or (3.13) it follows that $\sum_{i=1}^n \gamma^2(\lambda_n y, i) = o(1)$, hence

$$\sum_{i=1}^n \ln(1 + \gamma(\lambda_n y, i)) = \sum_{i=1}^n \gamma(\lambda_n y, i) + o(1). \tag{3.61}$$

Set

$$Q(x)=\sqrt{\left(y-\frac{bR_2(x)}{\lambda_n}\right)^2-4a^2\left(\frac{R_1(x)}{\lambda_n}\right)^2}, \tag{3.62}$$

where (3.47–(3.50) have been used. We find from (3.46), (3.22), and (3.8) that $|Q(x)|>C>0$ for $\alpha\geqslant 0$, $y\notin[A,B]$, $1\leqslant x\leqslant n$, for n sufficiently large. Applying the Euler–Maclaurin formula [16, p. 283] to the sum on the right hand side of (3.61) gives

$$\begin{aligned}\sum_{i=1}^{n} &\ln(1+\gamma(\lambda_n y, i))\\ &=\int_M^n \gamma(\lambda_n y, x)\,dx+\frac{1}{2}\gamma(\lambda_n y, M)+\frac{1}{2}\gamma(\lambda_n y, n)\\ &\quad+\sum_{i=1}^{M-1}\gamma(\lambda_n y, i)+\frac{1}{2}\int_M^n (B_2-B_2(x-[x]))\frac{d^2}{dx^2}\gamma(\lambda_n y, x)\,dx+o(1).\end{aligned} \tag{3.63}$$

Here

$$\gamma(\lambda_n y, x)=\frac{(b(R_2(x+1)-R_2(x)))/\lambda_n+Q(x)-Q(x+1)}{2Q(x)} \tag{3.64}$$

which is finite since $|Q(x)|>0$. Equation (3.30) implies that the second and third terms on the right hand side of (3.63) are $o(1)$. For fixed M the same is true for the fourth term on the right hand side of (3.63). If we differentiate (3.64) with respect to x we find

$$\begin{aligned}\gamma(\lambda_n y, x)'&=\frac{b((R_2'(x+1)-R_2'(x)/\lambda_n)+Q'(x)-Q'(x+1)}{2Q(x)}\\ &\quad-\left\{\frac{b((R_2(x+1)-R_2(x)/\lambda_n)+Q(x)-Q(x+1)}{2Q^2(x)}\right\}Q'(x).\end{aligned}$$

Differentiating once again, then using Lemma 3.7 and the bound $|B_2-B_2(x)|\leqslant\frac{1}{4}$ for $x\in[0,1]$, yields

$$\begin{aligned}&\left|\int_M^n (B_2-B_2(x-[x]))\,\gamma''(\lambda_n y, x)\,dx\right|\\ &\qquad\leqslant\frac{1}{4}\int_M^n |\gamma''(\lambda_n y, x)|\,dx\\ &\qquad=O\left[\int_M^n \frac{|R_2''(x)|}{\lambda_n}+\left(\frac{|R_1'(x)|+|R_2'(x)|}{\lambda_n}\right)^2 dx\right]\end{aligned}$$

which is $o(1)$ by Lemma 3.7. Therefore

$$\sum_{i=1}^{n} \ln(1+\gamma(\lambda_n y, x)) = \int_M^n \gamma(\lambda_n y, x)\, dx + o(1). \tag{3.65}$$

An application of Taylor's formula with remainder to (3.64) yields

$$\gamma(\lambda_n y, x) = \frac{(b/\lambda_n)(R_2'(x) + \frac{1}{2}\int_x^{x+1} R_2''(z)(x+1-z)\, dz)}{2Q(x)} - \frac{\int_x^{x+1} Q'(z)\, dz}{2Q(x)}.$$

The numerator of the second term on the right hand side of the above equation can be rewritten as

$$\int_x^{x+1} Q'(z)\, dz = Q'(x) + \frac{1}{2}\int_x^{x+1} Q''(z)(x+1-z)\, dz.$$

From Lemma 3.7 we find

$$\frac{1}{\lambda_n}\int_M^n \left|\frac{\int_x^{x+1} R_2''(z)\, dz}{2Q(x)}\right| dx \leqslant \frac{1}{\lambda_n}\int_M^n \frac{\sup_{z\in[x,x+1]} |R_2''(z)|}{2\,|Q(x)|}\, dx = o(1).$$

Now

$$Q'(x) = \left(-b\left(y - \frac{bR_2(x)}{\lambda_n}\right)\frac{R_2'(x)}{\lambda_n} - 4a^2\,\frac{R_1(x)}{\lambda_n}\,\frac{R_1'(x)}{\lambda_n}\right)\Big/ Q(x),$$

and it follows from Lemma 3.7 that

$$\begin{aligned}\int_M^n &\left|\frac{\int_x^{x+1} Q''(z)(x+1-z)\, dz}{Q(x)}\right| dx \\ &= O\left(\int_M^n \int_x^{x+1} \{a\,|R_1''(x)| + b\,|R_2''(x)| + a^2\,|R_1'(x)|^2 + b^2\,|R_2'(x)|^2\}\, dx\right) \\ &= o(1).\end{aligned}$$

These results imply that

$$\int_M^n \gamma(\lambda_n y, x)\, dx = \frac{b}{\lambda_n}\int_M^n \frac{R_2'(x)}{2Q(x)}\, dx - \int_M^n \frac{Q'(x)}{2Q(x)}\, dx + o(1). \tag{3.66}$$

If we integrate the second term on the right hand side of the above equation, then let n tend to infinity, we find

$$\lim_{n\to\infty} \int_M^n \frac{Q'(x)}{2Q(x)}\, dx = \lim_{n\to\infty} \frac{1}{2}\ln\frac{Q(n)}{Q(M)} = \frac{1}{2}\ln\frac{\sqrt{(y-b)^2 - 4a^2}}{y},$$

since (3.2) and Lemma 3.7 show that $\lim_{n\to\infty} R_2(n)/\lambda_n = \lim_{n\to\infty} R_1(n)/\lambda_n = 1$. Choose $\varepsilon > 0$ then fix the integer M sufficiently large so that $|R_1(x)/R_2(x) - 1| < \varepsilon$ for all $x \geqslant M$, and examine the first integral on the RHS of (3.66),

$$\left|\frac{1}{\lambda_n}\int_M^n \frac{R_2'(x)\,dx}{Q(x)}\,dx - \frac{1}{\lambda_n}\int_M^n \frac{R_2'(x)}{Q_2(x)}\,dx\right| = \frac{4a^2}{\lambda_n}\left|\int_M^n \frac{R_2'(x)((R_1(x)/R_2(x))^2-1)\,R_2^2(x)/\lambda_n^2}{Q(x)\,Q_2(x)(Q(x)+Q_2(x))}\,dx\right|. \tag{3.67}$$

Here

$$Q_2(x) = \sqrt{\left(y - \frac{bR_2(x)}{\lambda_n}\right)^2 - 4a^2\left(\frac{R_2(x)}{\lambda_n}\right)^2}.$$

From (3.46), (3.22), and (3.8) there exists a $C > 0$ such that $|Q_2(x)| > C$, for $y \notin [A, B]$, and $0 \leqslant x \leqslant n$, n sufficiently large. Furthermore $Q(x) \neq -Q_2(x)$ for n large enough which implies that there exists a $\hat{C} > 0$ such that $|Q(x) + Q_2(x)| > \hat{C}$. These inequalities show that the integral on the RHS of (3.67) is bounded by

$$\leqslant \frac{4a^2\varepsilon}{3\lambda_n^3 C^2 \hat{C}}\int_M^n |dR_2^3(x)| \leqslant \frac{4a^2\varepsilon}{3C^2\hat{C}}\,\frac{R_2^3(n)}{\lambda_n^3}, \tag{3.68}$$

where the change of variable $z = R_2(x)/R_2(n)$ has been used to perform the integration. The same substitution shows

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{\lambda_n}\int_M^n \frac{R_2'(x)}{Q_2(x)}\,dx \\
&= \lim_{n\to\infty}\frac{R_2(n)}{\lambda_n}\int_{R_2(M)/R_2(n)}^1 \frac{dz}{\sqrt{(y - bz(R_2(n)/\lambda_n))^2 - 4a^2(R_2^2(n)/\lambda_n^2)\,z^2}} \\
&= \int_0^1 \frac{dz}{\sqrt{(y-bz)^2 - 4a^2z^2}},
\end{aligned} \tag{3.69}$$

where (3.2) and Lemma 3.7 have been used to obtain the last equality. Therefore (3.69), (3.68), and (3.66) yield

$$\lim_{n\to\infty}\left|\sum_{i=1}^n \ln(1+\gamma(\lambda_n y, x)) - \frac{b}{2}\int_0^1 \frac{dz}{\sqrt{(y-bz)^2-4a^2z^2}} + \frac{1}{2}\ln\sqrt{\frac{(y-b)^2-4a^2}{y^2}}\right| \leqslant \frac{4a^2\varepsilon}{C^2\hat{C}}$$

and since ε was arbitrary this gives the result.

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